ON THE ALLOCATION OF NEW INPUTS AND OUPUTS WITH DEA

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Abstract: We propose a new interpretation to the solution of the problem of allocating a new total fixed input or output to a set of Decision Making Units (DMU's) under the assumption that a fair way to do it is assuring that all DMU's are placed on the efficiency frontier.

Key Words: DEA; Total Fixed Input Allocation; Total Fixed Output Allocation; Parametric DEA

Resumo: Propomos uma nova interpretação para a solução do problema de alocação de um novo insumo ou produto cujo valor total seja fixo a um conjunto de Unidades Tomadoras de Decisão (DMU's) sob a hipótese de que um jeito justo de fazê-lo seja assegurando que todas as DMU's sejam colocadas sobre a fronteira de eficiência.

Palavras-chave: DEA; Alocação de insumos com total fixo; Alocação de produtos com total fixo; DEA paramétrico

I. INTRODUCTION

A Bank intends to introduce a new product and has a global sales goal for the first year that wishes to share among its branches. A University receives a donation free of use specification and wants to distribute it among its departments.

Problems of this nature, in which a new total fixed output (as in the first case) or input (second case) ought to be shared among several corporation units are very common and have recently been the focus of attention of people who work with decision making tools, in particular those who work with Data Envelopment Analysis.

Cook and Kress (1999) were the first authors to treat the problem of allocating a new total fixed input within a DEA formulation. The model proposed by them was based on the idea that the DMU (Decision Making Units) efficiencies should remain constant after the allocation of the new input. The results obtained were later improved by Cook and Zhu (2005).

Beasley's (2003) approach had a different assumption. According to the model he proposed, the allocation of a new input should be such that, at the end, all DMU's should be efficient (or at least technically efficient). The same assumption was adopted by Gomes and Estellita Lins (2008), Avellar et al. (2007), Guedes et al. (2008) and Milioni et al. (2008), who worked, in the last 3 cases, with the so called parametric DEA formulation, characterized by the fact that the efficiency frontier obeys a specific and predefined locus of points.

Other works using DEA with different approaches upon the same problem are Takeda (2000), Wei et al (2000), Yan et al (2002), Jah-

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anshahloo et al. (2003), Korhonen and Syrjänen (2004) and Soares de Mello et Al. (2006).

The assumption that the allocation of a new input should be such that at the end all DMU's become efficient generated an interpretation problem on the results obtained which, for sometime, has troubled those who work in the area. It is not difficult to understand it.

Suppose one wants to allocate a new total fixed input of 100 units to a set of DMU's. Suppose that the allocation using Beasley's method, for instance, assigns 27.4 units of that input to a specific DMU *k*. As expected by Beasley's formulation, at the end of the allocation process all DMU's are placed on the efficiency frontier.

Suppose, however, that a scale error was found on the total value of the new input which should be equal to 10, and not 100. Then, the allocation previously made is disregarded and a new allocation is made using the same method. In this case, without surprises, considering that Beasley's method deals with constant returns to scale, DMU *k* which had received 27.4 units of the total 100 would now receive 2.74 units of the corrected total 10. And again, just like all other DMU's, DMU k would be located on the efficiency frontier.

The problem now should be clear. It is equally possible to make all DMU's efficient through the distribution of a new total fixed input whose value is equal to 100, or 10. Actually, with the same rationale, it would be equally possible to do it in case the total fixed input were equal to 1, or to 0.1, or 0.01…

The intriguing question is that, apparently, it is possible to make all DMUs efficient just by distributing an insignificant amount of a certain input amongst them. Does this make any sense, or is this an evidence of a logical mistake in the process? This question has annoyed researchers and has been the object of many discussions in symposiums, workshops and even in some dissertation committees.

The main goal of this paper is to propose an answer to this question.

II. INTERPRETATION

It becomes easier to present and to understand the solution of this problem by making an analogy with a linear regression problem.

Consider the hypothetical data presented in Table 1. The result of the regression of *Y* on *X* is the equation $Y = 20.5 + 8.1X$ which has a coefficient of determination of *0.73*. The regression residuals are also presented in Table 1.

Table 1: Data, Regression Residuals

Case j	*Y(j)*	*X(j)*	Residuals *R(j)*
1	58	5	3.1
2	89	8	-3.6
3	109	12	8.8
4	135	14	-0.9
5	111	11	-1.3
6	98	11	11.7
7	122	8	-36.6
8	64	7	13.3
9	99	9	-5.5
10	50	5	11.1

Let now *Min* and *Max* be the minimum and maximum values of *R(j)*, respectively. Define $R'(j) = [R(j) - Min] / [Max - Min], j=1,2...,10.$ Notice that the values of the vector *R'* vary from 0 to 1, and that they are proportional to the distances of *R(j)* to *Min*. Let now *S* be the sum of all *R'(j)* and define $R''(j) = R'(j) / S$. Then, the sum of all *R"(j)* will be naturally equal to 1, and its values will maintain the proportionality to the distances of *R(j)* to *Min*. *R*, *R'* and *R"* values are presented in Table 2. Since *R"* is a linear transformation of *R*, the regression of *Y* on *X* and *R"* will be statistically perfect, in the sense that it will have a coefficient of determination of *1* and all the residuals will be null (Gujarati, 2004).

Table 2: Residuals *R* and linearly transformed residuals *R'* e *R"*

Case j	*R(j)*	*R'(j)*	*R"(j)*
1	3.1	0.80	0.11
2	-3.6	0.66	0.09
3	8.8	0.91	0.12
4	-0.9	0.72	0.10
5	-1.3	0.71	0.10
6	11.7	0.97	0.13
7	-36.6	0.00	0.00
8	13.3	1.00	0.14
9	-5.5	0.62	0.08
10	11.1	0.96	0.13

Suppose now that *Y* represents the production of each of 10 production units or DMU's and that *X* is their respective number of employees. From the regression estimated slope coefficient we have that the average production per employee is equal to 8.1 production units. Assume that the corporation to which all DMU's belong wishes to share among them a new input *Z* whose total value is equal to 100 (e.g., 100 additional units of energy). A possible "fair" way to do it would be to allocate *Z(j)* to DMU *j* such that *Z(j)* = 100.*R"(j)*. The "fairness" of the act would be settled upon the fact that larger shares would be allocated to more productive DMU's (i.e., larger distances from *R(j)* to *Min*).

For the reasons that have already been pointed out, in this case, a regression of *Y* on *X* and *Z* would be statistically perfect.

Now, call **R** the set of real numbers. The initial problem in which *X* and *Y* were the only variables was defined in $\mathbf{R}^2$, while the problem which includes *Z* is defined in $\mathbf{R}^3$. Notice, however, that the statistically perfect solution in $\mathbf{R}^3$ does not change anything on the solution of the initial problem in $\mathbf{R}^2$. Thus, it is not surprising that the statistically perfect solution in $\mathbf{R}^3$ could be achieved regardless of the total value or even the meaning of the new input *Z*. Indeed, if fractions *F(j)* of a single unit of energy were allocated to the DMU's according to the proportions observed in *R"(j)*, i.e., *F(j)=R"(j)*, the regression of *Y* on *X* and the fractions *F(j)* in $\mathbf{R}^3$ would be statistically perfect, as well.

The regression problem described above is completely analogous to the problem of allocating a new input using DEA models such as the ones proposed by Beasley (2003), Avellar et al. (2007), Guedes et al. (2008) and Milioni et al. (2008). In each one of them, allocating a new total fixed input is the result of different methods with different degrees of complexity sharing, however, two basic features: (i) the final result is expressed as percentages of the new total fixed input that should be allocated to each DMU and (ii) just as in the regression problem, the total value and even the meaning of this new total fixed input are irrelevant to the solution of the problem.

The key point to understand why this is so is to realize that making a "fair" allocation of a new input is equivalent to adding a new dimension to the problem. Call *m* and *s* the total number of inputs and outputs, respectively, and notice that before the allocation of the new input the DEA problem is defined in $\mathbf{R}^{m+s}$, whereas after the allocation of the new input all DMU's will be placed over an efficient frontier defined in $\mathbf{R}^{m+s+1}$. Just like in the regression problem illustrated above, however, the efficiency frontier in $\mathbf{R}^{m+s+1}$ does not change anything on the initial problem defined in $\mathbf{R}^{m+s}$.

Therefore, "fairly" allocating a new input should not be mistaken with making the whole scenario "fair". It is not surprising, thus, that the problem can be solved regardless of the new input magnitude and relevance, for – and this is very important – solving the problem does not immediately legitimize the inclusion of this new input in the set of inputs and outputs that characterizes the DMU's concerning the problem of evaluating their relative efficiencies.

Hence, just like in the regression problem, there would be nothing odd with the fact that if one of the DEA methods listed above were used to allocate slices of an apple – a purposely ludicrous proposition – to a set of DMU's, at the end of the allocation all DMU's would be placed on the efficiency frontier in the augmented space of inputs and outputs This does not mean, however, that it becomes justified the inclusion of this last input – slices of an apple – in the list of relevant inputs and outputs for the purpose of relative efficiency evaluation of the DMU's.

III. CONCLUSION

As stated earlier, we proposed a new interpretation to the solution of the problem of allocating a new total fixed input or output to a set of DMU's under the assumption that a fair way to do it is assuring that all DMU's become efficient.

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